3. Array Calculation - Density Analogue Method

Another commonly used method for determining the size of cubic arrays of identical subcritical units is called the density analogue method.(7)

Density analogue is based upon the relationship of a bare spherical critical mass, $M_{c,b}$, and the density of the fissile material, ρ, or:

$$M_{c,b} \propto (\rho)^{-2} \qquad (a)$$

$$\frac{M_{c,b}}{M_{co,b}} = \left(\frac{\rho}{\rho_o}\right)^{-2} \qquad (b)$$

Where $M_{co,b}$ is the bare spherical critical mass at a different density, ρ_o.

Since we usually deal with shapes other than spheres, the exponent, 2, is replaced with "S", that can be no greater than 2. The exponent "S" is a function of the size, shape, and nuclear properties of the fissile material as well as any reflecting material near the system.

We usually deal with large arrays of units where each unit is much less than half of a critical mass. Since the effect of reflection on S is not readily available for most systems, bare arrays are calculated and conservative reflection and interspersed moderation factors are applied to the bare array results. For bare arrays S can be approximated by:

$$S = 2(1-f) \qquad (c)$$

where

$$f = \frac{M_{e,b,s}}{M_{c,b}}, \text{ the fraction of the critical bare spherical mass of the unit}$$

$M_{e,b,s}$, the mass of the bare sphere equivalent to the mass in the geometry being studied, may be determined by equating spherical buckling to the buckling of the geometry in question and solving for the sphere radius as:

For a cylinder,

$$\frac{\pi^2}{(R_{sp} + \lambda)^2} = \frac{J_o^2}{(R_{cy} + \lambda)^2} + \frac{\pi^2}{(H_{cy} + 2\lambda)^2} \qquad (d)$$

For a cube or parallelepiped,

$$\frac{\pi^2}{(R_{sp}+\lambda)^2} = \frac{\pi^2}{(a+2\lambda)^2} + \frac{\pi^2}{(b+2\lambda)^2} + \frac{\pi^2}{(c+2\lambda)^2} \quad \text{(e)}$$

The inverse ratio of densities in equation (b) can become the ratio of the volumes since the masses of fissile material in identical units are equal. Equation (b) then becomes:

$$M_{co,b} = M_{c,b}\left(\frac{V_{cell}}{V_{unit}}\right)^S \quad \text{(f)}$$

dividing by M_e, the equivalent mass of the units,

$$N_c = \frac{M_{c,b}}{M_e}\left(\frac{V_{cell}}{V_{unit}}\right)^S \quad \text{(g)}$$

where

N_c is the number of units necessary for a critical bare array.

To obtain the fully reflected array size, the bare array is reduced by the reflection factor found in Figure V.D.1-4. In this figure the array reflection factor varies with the material in the units (i.e., the hydrogen atom to fissile atom ratio). In reality, this factor also varies with unit size, the average fissile material density, and the reflector material and thickness.(13) For this reason care must be exercised in applying these factors to array calculations other than density analogue. Density analogue calculations of experimental metal and solution arrays have given conservative results when this factor has been used.

Two of the points in Figure V.D.1-4, as shown, have been determined experimentally for small arrays of U-235 metal and uranyl nitrate solutions of an $H/^{235}U$ of 59. The curves are extended by calculational data.(7) A reflection factor of 20 for plutonium metal has been calculated by D. R. Smith of the Los Alamos Scientific Laboratory. The plutonium reflection factor curve is based upon the Pu/U metal ratio (20/13) and extended to other H/X ratios. This is probably overly conservative for the higher H/Pu ratios.

Example of Density Analogue Correlation

Calculate an experimental square pitch cubic critical array of 64 right-circular cylinders of uranyl (92.6 Wt% U-235) nitrate solution (415 g U/l), sp.gr. 1.555.(8)

Containers: Lucite, 20.32 cm O.D. and 18.84* cm outside height, wall thickness 0.64 cm. Surface-to-surface separation of units at critical was 10.67 cm.

*The cylinders were filled to exactly 5.000 liters ± 0.5 g sol. giving this calculated solution height. The outside height of the containers was actually 19.05 cm.

From page III.B.10(93)-1 the material buckling of 415 g U/l UNH is 0.03020 cm^{-2} and the bare extrapolation distance λ_b, is 2.11 cm. The critical, bare, spherical mass at this concentration is calculated from this data.

$$R_{sp} = \sqrt{\frac{\pi^2}{B_g^2}} - \lambda_b = \frac{3.1416}{0.1738} - 2.11 = 15.97 \text{ cm}$$

$$\text{Vol.sp} = 0.004189\,(15.97)^3 = 17.062 \text{ liters}$$

$$M_{c,b} = (17.062)(415 \text{ g U/l})(0.926) = 6{,}557 \text{ g } ^{235}U$$

$$M_e = \text{mass of unit} = (5)(384.38\ ^{235}U/l) = 1{,}921 \text{ g } ^{235}U$$

$M_{e,b,s}$, the mass of a bare sphere equivalent to the mass in the shape being considered may be determined by equating spherical buckling to the shape buckling as:

$$\frac{\pi^2}{(R_{sp} + \lambda)^2} = \frac{J_o^2}{(R_{cy} + \lambda)^2} + \frac{\pi^2}{(H_{cy} + 2\lambda)^2} \quad \text{all dimensions are in cm.}$$

For this experiment,

$$R_{cy} = 9.52 \text{ cm}$$

$$H_{cy} = 17.561 \text{ cm}$$

λ_b = 2.11 However, the 0.64 cm wall increased the extrapolation length by approximately 0.8 cm (see page II.E-5). Reference LA-3612 indicates plexiglas <1.0 cm is equivalent to polyethylene.

$$\therefore \lambda = 2.11 + 0.8 = 2.91$$

$$\frac{9.87}{(R_{sp} + 2.91)^2} = \frac{5.784}{(9.52 + 2.91)^2} + \frac{9.87}{(17.561 + 5.82)^2}$$

$$= \frac{5.784}{154.5} + \frac{9.87}{546.7} = .037436 + .018054 = .055491$$

$$R_{sp} = \frac{\pi}{\sqrt{.05549}} - \lambda = 13.336 - 2.91 = 10.426 \text{ cm}$$

$$V_{sp} = (.004189)(10.426)^3 = 4.748 \text{ liters}$$

$$M_{e,b,s} = (4.748)(384.3) = 1{,}824 \text{ g } ^{235}U$$

and

$$S = 2\left(1 - \frac{1824}{6557}\right) = 1.433$$

$$V_{cell} = (10.67 + 20.32)^2(10.67 + 19.05)\ 10^{-3} = 28.542 \text{ liters}$$

$$V_{unit} = 5.0 \text{ liters}$$

$$N_c = \frac{M_{c,b}}{M_e} \left(\frac{V_{cell}}{V_{unit}}\right)^S$$

$$= \frac{6557}{1921} \left(\frac{28.542}{5.0}\right)^{1.443}$$

$$= 3.413\ (12.3495)$$

$$= 42$$

Or compared to the actual critical number of 64, density analogue is conservative by 35 percent. A comparison of experiment with the density analogue method gave the numbers (Table V) for other bare critical arrays of the same containers and materials as used in the example.

TABLE V

Five Liter U(92.6)NH Equilateral Cylinder Arrays (8)

Cubic Array	Surface-to-Surface, cm	Number of Units Critical: Experiment	Number of Units Critical: Calculated	GEM-III K_{eff}
2 x 2 x 2	1.43	8	8.9	
3 x 3 x 3	6.48	27	23	
4 x 4 x 4	10.67	64	42	0.953
5 x 5 x 5	14.40	125	69	

Note that the 2 x 2 x 2, close array is nonconservative as well as the GEM-III calculations on the 4 x 4 x 4 array.

The density analogue method was also used to calculate the close packed, long U(92.6)NH bottle experiment used in the solid angle example (see Table IV and Table VI, pages V.B.2-8 and V.B.3-5).

Density analogue appears to be nonconservative for single tier arrays of long bottles, but when the bottles are stacked and the array more closely approaches a cube, the results are conservative. This may be better shown in Figure V.D.1-5, where it appears that the density analogue method is conservative when the bottles are stacked two or more tiers high or for a large single tier where their surface-to-surface spacing is greater than 8 inches. Care must be exercised when using this method to calculate safe tall cylinder arrays.

TABLE VI

Density Analogue Results for 410 g U/l, U(92.6)NH in 5.375" O.D., 12.76 Liter Bottles (9)

Single Tier Square Array	Surface-to-Surface, in.	Number of Units Critical Experiment	Number of Units Critical Calculated	GEM-III K_{eff}
3 x 3	1.75	9	8.5	.9883
4 x 4	3.32	16	17.6	
5 x 5	4.55	25	28.7	
6 x 6	5.64	36	41.7	.9451
9 x 9	7.79	81	82	
Double Tier				
4 x 4	3.72	32	21	
5 x 5	5.35	50	38	
7 x 7	8.33	98	91	

Density analogue has been used quite extensively in calculating metal arrays. An example follows of the plutonium ingot array experiments carried out at the Lawrence Radiation Laboratory.(10)(11)

Data: A cubic array of 64 (4x4x4), 3.026 kg (19.6 g/cm^3) of 6.5 cm diameter and 4.6 cm high, with center-to-center horizontal spacing (x and y) of 12.513 cm and vertical spacing (z) of 7.858 cm, was critical. The bare spherical critical mass of plutonium is taken as 10.2 kilograms.

To obtain the buckling conversion from the cylinders to spheres, the bare extrapolation distance of plutonium metal is needed. This was obtained from DP-532 (12) pages 207 and 219 as 1.582 cm.

The buckling conversion is then

$$\frac{\pi^2}{(R_{sp} + \lambda_b)^2} = \frac{J_o^2}{(R_{cy} + \lambda_b)^2} + \frac{\pi^2}{(H_{cy} + 2\lambda_b)^2}$$

$$R_{sp} = \frac{\pi}{\left[\frac{J_o^2}{(R_{cy} + \lambda_b)^2} + \frac{\pi^2}{(H_{cy} + 2\lambda_b)^2}\right]^{1/2}} - \lambda_b$$

$$= \frac{\pi}{\left[\frac{5.784}{(3.25 + 1.582)^2} + \frac{9.87}{(4.6 + 3.164)^2}\right]^{1/2}} - 1.582$$

$$= 3.3156 \text{ cm}$$

$$\mathrm{Vol}_{sp} = 4.189(3.3156)^3 = \mathrm{Vol}_{unit}$$

$$M_e = 152.69(19.6\ \mathrm{g/cm^3}) = 2992.7\ \mathrm{g\ Pu}$$

$$S = 2\left(1 - \frac{2992.7}{10,200}\right) = 1.413$$

$$\mathrm{Vol}_{cell} = (12.513)^2(7.858) = 1230.37\ \mathrm{cm^3}$$

$$N_c = \frac{10.2}{2.9927}\left(\frac{1230.4}{152.7}\right)^{1.413} = 65$$

The result is slightly nonconservative by 1.6 percent. If no buckling conversion is made, the density analogue method gives a conservative result of 62.5 units critical.

The density analogue method can be used equally well for uranium metal arrays. Table VII lists some of the uranium and plutonium metal arrays calculated by density analogue. Each array was calculated by using the buckling conversion and also by using the shape allowance factor obtained from page II.B.4-1. The arrays were also calculated without applying a geometry correction. The uncorrected calculations yielded conservative results in all cases, 18 to 44 percent lower than the actual arrays of metal cylinders. However, for the plutonium arrays the calculated results were within 1.5 percent of the experimental numbers. Use of the shape allowance factors yielded nonconservative results in most cases and should not be used with density analogue.

TABLE VII

Density Analogue Calculations of Metal Critical Experiments

Uranium(93.2) Metal of Various Dimensions

Geometry*	Unit Mass, Kilograms	H/D	Array	Exp. No. Units	Calculated No. Units: Shape Cor.	B_g^2 Cor.	Uncorrected
A4	10.487	.948	3 x 3 x 3	27	29	22	22
A4	10.487	.948	4 x 4 x 4	64	68	51	50
A6	10.434	.47	4 x 4 x 4	64	100	61	42
C3	20.877	.94	3 x 3 x 3	27	21	17	15
B2	15.683	.70	3 x 3 x 3	27	26	21	18

Plutonium Metal (2.6" dia., 1.8" high in Al cans, 3.026 Kgs Pu)

S-to-S Separation		H/D	Array	Exp. No. Units	Calculated No. Units: Shape Cor.	B_g^2 Cor.	Uncorrected
x,y,z	0.75 cm	0.7	2 x 2 x 2	8	10	8.0	8
x,y,z	2.95 cm	0.7	3 x 3 x 3	27	36	27.3	27
x,y	12.513 cm	0.7	4 x 4 x 4	64	88	63.9	63
z	7.858 cm						

*See page V.B.4-2 for definition.